FIRST CRITICAL FIELD IN THE PINNED THREE-DIMENSIONAL GINZBURG–LANDAU MODEL: A MATCHING UPPER BOUND

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Abstract. We continue our study of the first critical field H_{c_1} for extreme type-II superconductors governed by the three-dimensional magnetic Ginzburg–Landau functional with a pinning term a_ε, as introduced in our previous work [DVR25]. Building upon the lower bound for H_{c_1} and the characterization of the Meissner solution, we now establish a matching upper bound for H_{c_1}, thereby identifying its leading-order behavior. This result confirms the sharpness of the previously derived lower bound and further elucidates the connection between the onset of vorticity and a weighted variant of the *isoflux problem*. Our argument is prompted by the upper bound construction we developed in [RSS25], based on the Biot–Savart law.

1. Introduction

The magnetic Ginzburg–Landau model [GL50] has long served as a cornerstone in the mathematical and physical study of superconductivity, providing a phenomenological framework that captures the macroscopic behavior of superconducting materials under applied magnetic fields. In type-II superconductors, the emergence of vortex filaments—localized regions where superconductivity breaks down—is a defining feature. These filaments nucleate when the applied magnetic field exceeds a threshold known as the first critical field, H_{c_1}. Understanding the precise onset of vorticity is a central problem, particularly in three dimensions, where vortex filaments are line-like and their geometry is more intricate, requiring the use of geometric measure theoretic tools in their analysis.

This work is the continuation of our study [DVR25] of the first critical field for the Ginzburg–Landau functional incorporating a spatially dependent pinning term $a_\varepsilon(x)$, which models material inhomogeneities. This setting reflects realistic superconducting samples where impurities or structural variations affect the energy landscape.

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After nondimensionalization of the physical constants, one may reduce to studying the energy functional

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = \frac{1}{2}\int_\Omega |\nabla_{\mathbf{A}}\mathbf{u}|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon(x)-|\mathbf{u}|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{H}-H_{\mathrm{ex}}|^2. \tag{1.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^3$, which we assume to be simply connected with C^2 boundary. The function $\mathbf{u}:\Omega\to\mathbb{C}$ is the *order parameter*; its squared modulus (the density of Cooper pairs of superconducting electrons in the BCS quantum theory [BCS57]) indicates the local state of the superconductor. The vector field $\mathbf{A}:\mathbb{R}^3\to\mathbb{R}^3$ is the electromagnetic vector potential of the induced magnetic field $\mathbf{H}=\operatorname{curl}\mathbf{A}$, and $\nabla_{\mathbf{A}}$ denotes the covariant gradient $\nabla - i\mathbf{A}$. The external (or applied) magnetic field is given by $H_{\mathrm{ex}}:\mathbb{R}^3\to\mathbb{R}^3$, which we assume takes the form $H_{\mathrm{ex}} = h_{\mathrm{ex}}H_{0,\mathrm{ex}}$, where $H_{0,\mathrm{ex}}$ is a fixed vector field and h_{ex} is a tunable real parameter representing the intensity of the external field. The parameter $\varepsilon>0$ is the inverse of the *Ginzburg–Landau parameter*, usually denoted κ, a non-dimensional constant depending only on the material. We are interested in the regime of small ε, corresponding to extreme type-II superconductors. Finally, a_ε is a function that accounts for inhomogeneities in the material; we assume $a_\varepsilon\in L^\infty(\Omega)$ and that it takes values in $[b,1]$, where $b\in(0,1)$ is a constant independent of ε. Regions where $a_\varepsilon=1$ correspond to sites without inhomogeneities.

We remark that the Ginzburg–Landau model is a $\mathbb{U}(1)$-gauge theory, in which all the meaningful physical quantities are invariant under the gauge transformations $\mathbf{u}\to\mathbf{u}e^{i\Phi}$, $\mathbf{A}\to\mathbf{A}+\nabla\Phi$, where Φ is any regular enough real-valued function. Two important gauge quantities are the supercurrent and the vorticity, respectively defined, for any sufficiently regular configuration $(\mathbf{u},\mathbf{A})$, as

$$j(\mathbf{u},\mathbf{A}) = (i\mathbf{u},\nabla_{\mathbf{A}}\mathbf{u}),\quad \mu(\mathbf{u},\mathbf{A}) = \operatorname{curl} j(\mathbf{u},\mathbf{A}) + \operatorname{curl}\mathbf{A}, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{C}$ identified with $\mathbb{R}^2$. As $\varepsilon\to 0$, the vorticity essentially concentrates in a sum of quantized Dirac masses supported on co-dimension 2 objects. This, together with the crucial fact that most of the energy concentrates around these objects, is what has essentially allowed mathematicians to analyze the model.

We refer the reader to [DVR25] and references therein for more background on the model. There, we established a lower bound for the first critical field H_{c_1} and characterized the Meissner solution, the unique (up to gauge transformation) vortexless minimizer of the energy below this threshold. Our analysis suggested that the onset of vorticity is governed by a weighted variant of the *isoflux problem* studied in [ABM06, Rom19a, RSS23, RSS25], hinting at a connection between the Ginzburg–Landau framework and geometric optimization, which is influenced by the presence of the pinning function. However, the absence of a matching upper bound left open the question of whether this optimization problem truly captures the leading-order behavior of H_{c_1}.

In the present work, we continue this investigation by establishing a matching upper bound for H_{c_1}, thereby identifying its leading-order behavior. This result confirms the sharpness of the lower bound derived in [DVR25] and completes the asymptotic characterization (to leading order) of the first critical field in the presence of pinning. Our argument is prompted by the upper bound construction we developed in [RSS25], which is based on the Biot–Savart law. This approach allowed us to capture not just the leading-order contribution of the (homogeneous) Ginzburg–Landau energy in the presence of prescribed bounded vorticity, but in fact the energy up to an $o_\varepsilon(1)$ error. It serves as a foundation for the present analysis in the weighted Ginzburg–Landau setting.

The upper bound in the present work is obtained by constructing a test configuration in which vorticity concentrates along smooth curves that are nearly optimal for the weighted isoflux problem. We show that the energy of this configuration becomes favorable once the applied field exceeds a threshold matching the lower bound provided in [DVR25], up to leading order. This confirms that the transition from the Meissner phase to the vortex phase is characterized by the weighted isoflux problem. Our analysis also reveals that the pinning term $a_\varepsilon(x)$ plays a subtle role in shaping the vortex landscape, influencing both the location and structure of the filaments.

Together with the results of [DVR25], our work provides a complete asymptotic description of the first critical field in the pinned three-dimensional Ginzburg–Landau model. This description aligns with the one we previously established for the analogous problem in the two-dimensional setting [DVR24], as well as in the homogeneous three-dimensional case [ABM06, BJOS13, Rom19a].

In order to state our first result, we recall that the analysis of (1.1) can be reduced to analyzing the weighted Ginzburg–Landau functional (see [DVR25, Section 2])

$$\widetilde{GL}_\varepsilon(u,A)=\frac12\int_\Omega \rho_\varepsilon^2|\nabla_A u|^2+\frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u|^2)^2+\frac12\int_{\mathbb{R}^3}|H-H_{\mathrm{ex}}|^2, \tag{1.3}$$

where the weight ρ_ε is the unique positive real-valued function that satisfies

$$\begin{cases} -\Delta\rho_\varepsilon &= \dfrac{\rho_\varepsilon}{\varepsilon^2}(a_\varepsilon-\rho_\varepsilon^2) \quad \text{in } \Omega\\ \dfrac{\partial\rho_\varepsilon}{\partial\nu} &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega.\end{cases}$$

We remark that $\mathbf{u}$ and u are related by $\mathbf{u}=\rho_\varepsilon u$, while $\mathbf{A}=A$. One straightforwardly verifies that $b\le\rho_\varepsilon^2\le 1$. We will assume throughout this article that a_ε is such that there exist $\alpha\in(0,1)$, $N>0$, and $C_1>0$ (that do not depend on ε) such that

$$\|\rho_\varepsilon\|_{C^{0,\alpha}(\Omega)}\le C_1|\log\varepsilon|^N. \tag{1.4}$$

Our first result establishes an upper bound for the free-energy functional associated with (1.3), that is, the functional considered in the absence of an external field. It provides an explicit configuration for which the vorticity concentrates on a given curve Γ parametrized by arc length. This construction is optimal at leading order and shows that the energetic cost of such a vortex line is, to leading order, bounded above by $\pi|\rho_\varepsilon^2\Gamma||\log\varepsilon|$, where

$$|\rho_\varepsilon^2\Gamma| := \int_0^{|\Gamma|} \rho_\varepsilon^2(\Gamma(s))\,\mathrm{d}s. \tag{1.5}$$

Theorem 1.1. *Let Γ be a C^2 simple open curve in Ω, parametrized by arc length, that intersects $\partial\Omega$ transversally. Assume (1.4) holds. Then, for any $\varepsilon > 0$ sufficiently small, there exists $(u_\varepsilon, A) \in H^1(\Omega,\mathbb{C}) \times H^1(\mathbb{R}^3,\mathbb{R}^3)$ such that*

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A) &:= \frac{1}{2}\int_\Omega \rho_\varepsilon^2|\nabla_A u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3}|\operatorname{curl} A|^2 \\ &\leq \pi|\rho_\varepsilon^2\Gamma||\log\varepsilon| + \left(\frac{N}{\alpha}+1\right)\pi\log|\log\varepsilon| + C_{\Omega,\Gamma} + o_\varepsilon(1), \end{aligned} \tag{1.6}$$

where $C_{\Omega,\Gamma}$ is the constant defined in (2.6).

In addition, it holds that, for any $\beta \in (0,1]$, we have

$$\|\mu(u_\varepsilon, A) - 2\pi\Gamma\|_{(C_T^{0,\beta}(\Omega,\mathbb{R}^3))^*} \leq C\varepsilon^{\frac{2}{3}\beta}|\log\varepsilon|^{1-\frac{2}{3}\beta}, \tag{1.7}$$

where $C_T^{0,\beta}(\Omega,\mathbb{R}^3)$ denotes the space of β-Hölder continuous vector fields defined in Ω whose tangential component vanishes on $\partial\Omega$.

We now turn to our main result on the first critical field. For that, we briefly recall two key ingredients from our previous work [DVR25]:

- *Meissner state:* The Ginzburg–Landau model admits a unique configuration, modulo gauge transformations, that we refer to as the Meissner state, in reference to the Meissner effect in physics. This state is obtained by minimizing the Ginzburg–Landau energy $GL_\varepsilon(\mathbf{u},\mathbf{A})$ under the constraint $|\mathbf{u}| = \rho_\varepsilon$. In the gauge where $\operatorname{div}\mathbf{A} = 0$ in $\mathbb{R}^3$, the Meissner state takes the form
$$(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0),$$
where ϕ_ε^0 and A_ε^0 depend on the domain Ω, the applied field $H_{0,\mathrm{ex}}$, and the weight function ρ_ε.

  Although this configuration is not a true critical point of (1.1), it provides a good approximation of the unique minimizer (still in the same gauge) below the first critical field, as $\varepsilon \to 0$.

  Closely related is a special vector field $B_\varepsilon^0 \in C_T^{0,1}(\Omega,\mathbb{R}^3)$, which satisfies
$$A_\varepsilon^0 - \nabla\phi_\varepsilon^0 = \frac{\operatorname{curl} B_\varepsilon^0}{\rho_\varepsilon^2} \quad \text{in } \Omega,$$
with $\operatorname{div} B_A = 0$ in Ω and $B_A \times \nu = 0$ on $\partial\Omega$.

  The regularity of the vector field B_ε^0, fundamental to our analysis, mainly depends on the regularity of $H_{0,\mathrm{ex}}$ in Ω. In particular, assuming henceforth $H_{0,\mathrm{ex}} \in L^3(\Omega,\mathbb{R}^3)$, we have that
$$\|B_\varepsilon^0\|_{C_T^{0,\gamma}(\Omega,\mathbb{R}^3)} \leq C \quad \text{for any } \gamma \in (0,1),$$

where the constant $C > 0$ does not depend on ε; see [DVR25, Proposition 2.2].

- *Weighted isoflux problem:* We let $\mathcal{N}(\Omega)$ be the space of normal 1-currents supported in $\overline{\Omega}$, with boundary supported on $\partial\Omega$. We denote by $|\cdot|$ the mass of a current. Recall that normal currents are currents with finite mass, whose boundaries have finite mass as well. We also let X denote the class of currents in $\mathcal{N}(\Omega)$ that are simple oriented Lipschitz curves. An element of X must either be a loop contained in $\overline{\Omega}$ or have its two endpoints on $\partial\Omega$.

  For any vector field $B \in C_T^{0,1}(\Omega, \mathbb{R}^3)$ and any $\Gamma \in \mathcal{N}(\Omega)$ we denote by $\langle \Gamma , B \rangle$ the value of Γ applied to B, which corresponds to the circulation of the vector field B on Γ when Γ is a curve.

  We also define $\rho_\varepsilon^2 \Gamma \in \mathcal{N}(\Omega)$ by duality, that is, for any 1-form ω supported in Ω, we let
$$\langle \rho_\varepsilon^2 \Gamma , \omega \rangle := \langle \Gamma , \rho_\varepsilon^2 \omega \rangle.$$

  Definition 1.1 (Weighted isoflux problem for type-II superconductivity)**.** *The weighted isoflux problem is the question of maximizing over $\mathcal{N}(\Omega)$ the ratio*
$$\mathrm{R}_{\rho_\varepsilon^2}(\Gamma) := \frac{\langle \Gamma , B_\varepsilon^0 \rangle}{|\rho_\varepsilon^2 \Gamma|}.$$

  Let us remark that the existence of maximizers for $\mathrm{R}_{\rho_\varepsilon^2}$ is ensured by the weak-$\star$ sequential compactness of $\mathcal{N}(\Omega)$. Also, let us notice that in the case Γ is a smooth curve parametrized by arc length, (1.5) holds.

In [DVR25] we showed that the occurrence of vortex filaments is possible only if $h_{\mathrm{ex}} \geq H_{c_1}^\varepsilon$, where
$$H_{c_1}^\varepsilon := \frac{|\log \varepsilon|}{2\mathrm{R}_\varepsilon} \quad \text{and} \quad \mathrm{R}_\varepsilon := \sup_{\Gamma \in X} \mathrm{R}_{\rho_\varepsilon^2}(\Gamma) = \sup_{\Gamma \in X} \frac{\langle \Gamma , B_\varepsilon^0 \rangle}{|\rho_\varepsilon^2 \Gamma|},$$
showing in particular that, for some constant $K_0 > 0$ independent of ε, one has
$$H_{c_1}^\varepsilon - K_0 \log |\log \varepsilon| \leq H_{c_1}.$$
Our main result on this paper complements this by providing an upper bound that matches this lower bound at leading order, that is,
$$H_{c_1} \leq H_{c_1}^\varepsilon + K^0 \log |\log \varepsilon|,$$
for some constant K^0 independent of ε.

We need the following assumptions. First, we assume that, for any ε sufficiently small, there exists a C^2 open simple curve Γ_ε parametrized by arc length which intersects $\partial\Omega$ transversely and such that
$$\mathrm{R}_{\rho_\varepsilon^2}(\Gamma_\varepsilon) = \mathrm{R}_\varepsilon + O_\varepsilon(|\log \varepsilon|^{-1}). \tag{1.8}$$
We assume in addition that there exists a positive constant C_0 (independent of ε) such that
$$|\Gamma_\varepsilon| \leq C_0 \tag{1.9}$$

and

$$C_{\Omega,\Gamma_\varepsilon} \le C_0 \log|\log\varepsilon|, \tag{1.10}$$

where $C_{\Omega,\Gamma_\varepsilon}$ is the constant defined in (2.6).

Theorem 1.2. *Assume (1.4), (1.8), (1.9), (1.10), and that* $\liminf\limits_{\varepsilon\to 0} \mathrm{R}_\varepsilon > 0$. *Then there exist* $\varepsilon_0 > 0$ *and* $K^0 > 0$ *such that, for any* $\varepsilon < \varepsilon_0$, *any* $\eta \in \left(0, \frac{1}{2}\right)$, *and any*

$$H_{c_1}^\varepsilon + K^0 \log|\log\varepsilon| \le h_{\mathrm{ex}} \le \varepsilon^{-\eta}, \tag{1.11}$$

the global minimizers $(\mathbf{u}, \mathbf{A})$ *of* GL_ε *do have vortices.*

The hypotheses (1.4), (1.8), (1.9), and (1.10) are assumed in order to have good control on the weighted isoflux problem as $\varepsilon \to 0$. In the special situation when $a_\varepsilon = a(x)$ is a C^1 function that does not depend on ε and such that it is constant near the boundary of Ω, when Ω is a ball, and $H_{0,\mathrm{ex}} = \hat{z}$, then all the hypotheses about Γ_ε are satisfied, provided that a is sufficiently close to 1. This ensures that the case remains comparable to the one analyzed for the (non-weighted) isoflux problem, see [ABM06, Rom19a, RSS23, RSS25]. In this case ρ_ε^2 converges uniformly to a as $\varepsilon \to 0$. We nonetheless expect the hypotheses above to hold in a much more general setting, which of course needs verification depending on the model of a_ε that one works with.

On the other hand, in [DVR25, Proposition 1.2], we provided a sufficient condition for $\liminf\limits_{\varepsilon\to 0} \mathrm{R}_\varepsilon > 0$ to hold.

Plan of the paper. The rest of the paper is organized as follows. In Section 2, we provide some preliminary results on tubular neighborhoods of a curve, the Biot–Savart vector field associated with a curve, and an energy splitting formula. In Section 3, we present a proof of Theorem 1.1. Finally, in Section 4, we provide a proof of Theorem 1.2.

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2. Some preliminaries

2.1. **Tubular neighborhood of a curve.** Let $\Gamma : [0, |\Gamma|] \to \mathbb{R}^3$ be a C^2 simple curve parametrized by arc length. We define $\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$ such that $(\Gamma'(s), \mathbf{e}_1(s), \mathbf{e}_2(s))$ is a direct orthonormal basis for any s and C^1 smooth with respect to s.

We define the regular tubular neighborhood $T_\delta(\Gamma)$ of Γ by

$$T_\delta(\Gamma) = \{\Gamma(s) + y \mid s \in [0, |\Gamma|],\ y \perp \Gamma'(s),\ |y| < \delta\},$$

where its thickness $\delta > 0$ is assumed to be sufficiently small so that the normal projection $\Pi : T_\delta(\Gamma) \to \Gamma([0, |\Gamma|])$ is well defined and C^1. In $T_\delta(\Gamma)$, we make use of curvilinear coordinates (s, v, w), that is

$$x = \Gamma(s) + x^\perp(s, v, w), \quad x^\perp(s, v, w) = v\mathbf{e}_1(s) + w\mathbf{e}_2(s), \tag{2.1}$$

for any $x = x(s, v, w) \in T_\delta(\Gamma)$. The volume element in these coordinates is

$$\mathrm{d}V = |1 - x^\perp \cdot \Gamma'| \mathrm{d}s\mathrm{d}v\mathrm{d}w = (1 + O(\delta))\mathrm{d}s\mathrm{d}v\mathrm{d}w. \tag{2.2}$$

2.2. **Biot–Savart vector field.** We next recall a result proved in [RSS25, Section 2], concerning the Biot–Savart vector field associated to a smooth simple closed curve Γ in $\mathbb{R}^3$, which is defined as

$$X_\Gamma(p) = \frac{1}{2} \int_t \frac{\Gamma(t) - p}{|\Gamma(t) - p|^3} \times \Gamma'(t)\, dt.$$

We recall that X_Γ is divergence-free, satisfies

$$\operatorname{curl} X_\Gamma = 2\pi\Gamma \quad \text{in } \mathbb{R}^3, \tag{2.3}$$

and belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for any $1 \leq p < 2$.

Moreover, if we let p_Γ denote the nearest point to p on Γ,

$$X_\Gamma(p) - \frac{p_\Gamma - p}{|p_\Gamma - p|^2} \times \Gamma'(p_\Gamma) \tag{2.4}$$

is in $L^q(T_\delta(\Gamma), \mathbb{R}^3)$, for any $q \geq 1$.

Proposition 2.1. *Assume Γ is a C^2 simple closed curve in $\mathbb{R}^3$ which intersects $\partial\Omega$ transversally. Then there exists a unique divergence-free $j_\Gamma : \Omega \to \mathbb{R}^3$, belonging to $L^p(\Omega, \mathbb{R}^3)$ for any $p < 2$, and a unique divergence-free $A_\Gamma \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ such that*

$$\begin{cases} \operatorname{curl}(j_\Gamma + A_\Gamma) &= 2\pi\Gamma & \text{in } \Omega \\ \nu \cdot j_\Gamma &= 0 & \text{on } \partial\Omega \end{cases}$$

and such that

$$-\Delta A_\Gamma = j_\Gamma \mathbf{1}_\Omega \quad \text{in } \mathbb{R}^3$$

holds in the sense of distributions. In particular, j_Γ and A_Γ only depend on $\Gamma \cap \Omega$. It also holds that $A_\Gamma \in W^{2,\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$, that $j_\Gamma - X_\Gamma \in W^{1,q}(\Omega, \mathbb{R}^3)$ for any $q < 4$, and that

$$j_\Gamma - X_\Gamma + A_\Gamma = \nabla f_\Gamma \quad \text{in } \Omega \tag{2.5}$$

for some harmonic function $f_\Gamma \in W^{1,q}(\Omega)$ for any $q < 4$.

An important role in the upper bound construction will be played by the following constant.

Definition 2.1. *Let Γ be a C^2 simple closed curve in $\mathbb{R}^3$ that intersects $\partial\Omega$ transversally. We apply Proposition 2.1 and define*

$$C_{\Omega,\Gamma} = \frac{1}{2} \int_{\mathbb{R}^3} |\operatorname{curl} A_\Gamma|^2 + \lim_{\rho \to 0} \left(\frac{1}{2} \int_{\Omega \setminus \overline{T_\rho(\Gamma)}} |j_\Gamma|^2 + \pi|\Gamma| \log \rho \right), \tag{2.6}$$

where $|\Gamma|$ denotes the length of Γ in Ω.

2.3. **Energy splitting.** Throughout this paper, we assume that $H_{\rm ex} \in L^2_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ is such that $\operatorname{div} H_{\rm ex} = 0$ in $\mathbb{R}^3$, consistent with the non-existence of magnetic monopoles. Consequently, there exists a vector potential $A_{\rm ex} \in H^1_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ such that

$$\operatorname{curl} A_{\rm ex} = H_{\rm ex} \quad \text{and} \quad \operatorname{div} A_{\rm ex} = 0 \text{ in } \mathbb{R}^3.$$

The natural space for minimizing GL_ε is $H^1(\Omega,\mathbb{C}) \times [A_{\rm ex} + H_{\rm curl}]$, where

$$H_{\rm curl} := \{\mathbf{A} \in H^1_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3) |\operatorname{curl} \mathbf{A} \in L^2(\mathbb{R}^3,\mathbb{R}^3)\};$$

see [Rom19a].

We next recall the energy splitting provided in [DVR25, Proposition 1.1], which plays a crucial role in the proof of Theorem 1.2.

Proposition 2.2. *Given any configuration* $(\mathbf{u},\mathbf{A}) \in H^1(\Omega,\mathbb{C})\times[A_{\rm ex}+H_{\rm curl}(\mathbb{R}^3,\mathbb{R}^3)]$*, letting* (u,A) *be defined via the relation* $(\mathbf{u},\mathbf{A}) = (\rho_\varepsilon u e^{ih_{\rm ex}\phi_\varepsilon}, A + h_{\rm ex}A^0_\varepsilon)$*, we have*

$$GL_\varepsilon(\mathbf{u},\mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\rm ex}\phi_\varepsilon}, h_{\rm ex}A^0_\varepsilon) + F_{\varepsilon,\rho_\varepsilon}(u,A) - h_{\rm ex}\int_\Omega \mu(u,A)\cdot B^0_\varepsilon + \mathfrak{r}, \tag{2.7}$$

where

$$\mathfrak{r} := \frac{h_{\rm ex}^2}{2}\int_\Omega \frac{|\operatorname{curl} B^0_\varepsilon|^2}{\rho_\varepsilon^2}(|u|^2-1). \tag{2.8}$$

3. Upper bound for the free energy

In this section we provide a proof for Theorem 1.1.

Proof of Theorem 1.1. We first let $r_\varepsilon := |\log\varepsilon|^{-q}$, for some $q > 1$ to be fixed later, and note that Γ can be extended to a C^2 simple closed curve in $\mathbb{R}^3$, with the extension being supported on the complement of $\overline{\Omega}$. Throughout this proof, although the curve Γ has been extended, its length is consistently measured with respect to its original definition within Ω.

We will construct a complex-valued function u_ε of the form

$$u_\varepsilon(x) := |u_\varepsilon|(x)e^{i\varphi(x)}.$$

Recalling Section 2.1, we start by defining its modulus as

$$|u_\varepsilon|(x) = \begin{cases} \dfrac{1}{f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)} f_0\left(\dfrac{\operatorname{dist}(x,\Gamma)}{\varepsilon}\right) & \text{if } x \in T_{r_\varepsilon}(\Gamma) \\ 1 & \text{if } x \in \Omega\setminus T_{r_\varepsilon}(\Gamma), \end{cases}$$

where hereafter f_0 denotes the modulus of the degree-one radial vortex solution u_0 that appears in [SS07, Proposition 3.11]. We recall from this book that $\lim\limits_{R\to\infty} f_0(R) \to 1$ and

$$\lim_{R\to\infty}\left(\frac{1}{2}\int_0^R\left(|{f_0}'|^2 + \frac{{f_0}^2}{r^2} + \frac{(1-{f_0}^2)^2}{2}\right)r\mathrm{d}r - \frac{1}{2\pi}\left(\pi\log R + \gamma\right)\right) = 0, \tag{3.1}$$

where $\gamma > 0$ is the universal constant first introduced in the seminal work [BBH94].

We now proceed to define the phase of u_ε. We let φ be defined via

$$\nabla\varphi = X_\Gamma + \nabla f_\Gamma \quad \text{in } \Omega, \tag{3.2}$$

where X_Γ and f_Γ are defined by applying Proposition 2.1. Let us remark that $\operatorname{curl} X_\Gamma = 2\pi\Gamma$ in $\mathbb{R}^3$ guaranties that φ is a well-defined function in Ω modulo 2π.

Finally, we apply once again Proposition 2.1, in order to define

$$A(x) = A_\Gamma(x).$$

The rest of the proof is divided in several steps.

Step 1. Estimating $|\nabla_A u_\varepsilon|^2 - |\nabla u_\varepsilon|^2$ over $T_{r_\varepsilon}(\Gamma)\cap\Omega$.

A straightforward computation shows that

$$|\nabla_A u_\varepsilon|^2 - |\nabla u_\varepsilon|^2 = |u_\varepsilon|^2\left(|A|^2 - 2\nabla\varphi\cdot A\right).$$

We denote hereafter $T^\Omega_{r_\varepsilon}(\Gamma) := T_{r_\varepsilon}(\Gamma)\cap\Omega$. Since $|u_\varepsilon|^2\le 1$, we apply Hölder's inequality to obtain

$$\left|\int_{T^\Omega_{r_\varepsilon}(\Gamma)} |\nabla_A u_\varepsilon|^2 - |\nabla u_\varepsilon|^2\right| \le \|A\|^2_{L^2(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)} + 2\|\nabla\varphi\|_{L^{\frac32}(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)}\|A\|_{L^3(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)}.$$

Since $A\in W^{2,\frac32}(\Omega,\mathbb{R}^3)$, from Sobolev embedding it follows that $A\in L^r(\Omega,\mathbb{R}^3)$ for any $r\ge 1$. In particular, from the Cauchy–Schwarz inequality, we obtain

$$\|A\|_{L^3(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)} \le \|A\|_{L^6(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)}|T^\Omega_{r_\varepsilon}(\Gamma)|^{\frac16} \le C r_\varepsilon^{\frac13} = o_\varepsilon(1).$$

Moreover, $\nabla\varphi\in L^r(\Omega,\mathbb{R}^3)$ for any $r<2$, which follows from Proposition 2.1. Hence,

$$\left|\int_{T^\Omega_{r_\varepsilon}(\Gamma)} |\nabla_A u_\varepsilon|^2 - |\nabla u_\varepsilon|^2\right| \le C\|A\|_{L^3(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)} = o_\varepsilon(1). \tag{3.3}$$

Step 2. Energy estimate over $T^\Omega_{r_\varepsilon}(\Gamma)$.

We claim that

$$\frac12\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2|\nabla u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \pi|\rho_\varepsilon^2\Gamma|\log\frac{r_\varepsilon}{\varepsilon} + \gamma|\Gamma| + o_\varepsilon(1), \tag{3.4}$$

where γ is the constant that appears in (3.1).

Let us start by observing that, since $\rho_\varepsilon\le 1$, we have

$$\frac12\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2|\nabla u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 \le \frac12\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2\left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right).$$

Next, given a point $p\in T_{r_\varepsilon}(\Gamma)$, we denote by p_Γ its nearest point on Γ and let

$$D_s := \{p\in T_{r_\varepsilon}(\Gamma)\ :\ p_\Gamma = \Gamma(s)\}.$$

Notice that D_s is a disk in $\mathbb{R}^2$ of radius r_ε centered at $\Gamma(z)$.

Recalling (2.4), we find that

$$h_\Gamma(p) := X_\Gamma(p) - Y_\Gamma(p) \in L^q\left(T_{r_\varepsilon}(\Gamma), \mathbb{R}^3\right)$$

for any $q \geq 1$, where

$$Y_\Gamma(p) := \frac{p_\Gamma - p}{|p_\Gamma - p|^2} \times \Gamma'(p_\Gamma).$$

Recalling (3.2) and applying Hölder's inequality, we deduce that

$$\int_{T^\Omega_{r_\varepsilon}(\Gamma)} |\nabla\varphi - Y_\Gamma|^2 = \int_{T^\Omega_{r_\varepsilon}(\Gamma)} |h_\Gamma + \nabla f_\Gamma|^2 \leq |T^\Omega_{r_\varepsilon}(\Gamma)|^{\frac{1}{3}} \, \|h_\Gamma + \nabla f_\Gamma\|^2_{L^3(T^\Omega_{r_\varepsilon}(\Gamma),\mathbb{R}^3)} \leq C r_\varepsilon^{\frac{2}{3}} = o_\varepsilon(1), \tag{3.5}$$

where we used the fact that $h_\Gamma + \nabla f_\Gamma \in L^3(T^\Omega_{r_\varepsilon}(\Gamma), \mathbb{R}^3)$.

Using (3.5), and recalling that $|u_\varepsilon| \leq 1$ and $\rho_\varepsilon \leq 1$, we deduce that

$$\begin{aligned}
&\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2 \left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \\
&\quad = \int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2 \left(|\nabla |u_\varepsilon||^2 + |u_\varepsilon|^2 |\nabla\varphi|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \\
&\quad = \int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2 \left(|\nabla |u_\varepsilon||^2 + |u_\varepsilon|^2 |Y_\Gamma|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) + o_\varepsilon(1).
\end{aligned} \tag{3.6}$$

Using the coordinates defined in (2.1), and recalling (2.2), we then find

$$\begin{aligned}
&\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2 \left(|\nabla |u_\varepsilon||^2 + |u_\varepsilon|^2 |Y_\Gamma|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \\
&\quad = \int_0^{|\Gamma|} \left(\int_{D_s\cap\Omega} \rho_\varepsilon^2 \left(|\nabla |u_\varepsilon||^2 + |u_\varepsilon|^2 |Y_\Gamma|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)(1+O_\varepsilon(r_\varepsilon)) \mathrm{d}v\mathrm{d}w\right) \mathrm{d}s.
\end{aligned}$$

Combining this with (3.6), we are led to

$$\begin{aligned}
&\int_{T^\Omega_{r_\varepsilon}(\Gamma)} \rho_\varepsilon^2 \left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \\
&\leq (1+O_\varepsilon(r_\varepsilon)) \int_0^{|\Gamma|} \left(\int_{D_s\cap\Omega} \rho_\varepsilon^2 \left(|\nabla |u_\varepsilon||^2 + |u_\varepsilon|^2 |Y_\Gamma|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) \mathrm{d}v\mathrm{d}w\right) \mathrm{d}s + o_\varepsilon(1).
\end{aligned} \tag{3.7}$$

Let us now deal with the weight. From the hypothesis (1.4) on ρ_ε, we observe that for any point $x = x(v,w,s) \in D_s \cap \Omega$, we have, using once again that $\rho_\varepsilon \leq 1$,

$$\begin{aligned}
|\rho_\varepsilon^2(x) - \rho_\varepsilon^2(\Gamma(s))| &= |\rho_\varepsilon(x) - \rho_\varepsilon(\Gamma(s))||\rho_\varepsilon(x) + \rho_\varepsilon(\Gamma(s))| \\
&\leq 2 r_\varepsilon^\alpha \|\rho_\varepsilon\|_{C^{0,\alpha}(T^\Omega_{r_\varepsilon}(\Gamma))} \leq 2C_1 r_\varepsilon^\alpha |\log\varepsilon|^N.
\end{aligned}$$

In particular, for any $q > \frac{N}{\alpha}$, we deduce that the RHS is $o_\varepsilon(1)$. Hence, using that $D_s\cap\Omega \subset D_s$, we find

$$\begin{aligned}(3.8)\quad &\int_0^{|\Gamma|}\left(\int_{D_s\cap\Omega}\rho_\varepsilon^2\left(|\nabla|u_\varepsilon||^2+|u_\varepsilon|^2|Y_\Gamma|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\mathrm{d}v\mathrm{d}w\right)\mathrm{d}s\\ &\leq \int_0^{|\Gamma|}\rho_\varepsilon^2(\Gamma(s))\left(\int_{D_s}\left(|\nabla|u_\varepsilon||^2+|u_\varepsilon|^2|Y_\Gamma|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\mathrm{d}v\mathrm{d}w\right)\mathrm{d}s+o_\varepsilon(1).\end{aligned}$$

Notice that we conveniently defined $|u_\varepsilon|$ slightly outside Ω, so that we do not have to perform a special computation next depending on the endpoints of Γ, viewed as the original curve without extension beyond $\overline{\Omega}$.

To compute the integral over D_s we use polar coordinates (r,θ) centered at $\Gamma(s)$. We have

$$|\nabla|u_\varepsilon|(x)| = \frac{\left|{f_0}'\left(\frac{r}{\varepsilon}\right)\right|}{\varepsilon f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)}|\nabla r| = \frac{\left|{f_0}'\left(\frac{r}{\varepsilon}\right)\right|}{\varepsilon f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)} \quad \text{and} \quad |Y_\Gamma| = \frac{r}{r^2}|\Gamma'(s)| = \frac{1}{r}.$$

It follows that

$$\begin{aligned}&\int_{D_s}\left(|\nabla|u_\varepsilon||^2+|u_\varepsilon|^2|Y_\Gamma|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\mathrm{d}v\mathrm{d}w\\ &\qquad = 2\pi\int_0^{r_\varepsilon}\left(\frac{{f_0}'\left(\frac{r}{\varepsilon}\right)^2}{\varepsilon^2 f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}+\frac{f_0\left(\frac{r}{\varepsilon}\right)^2}{r^2 f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}+\frac{1}{2\varepsilon^2}\left(1-\frac{f_0\left(\frac{r}{\varepsilon}\right)^2}{f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}\right)^2\right)r\mathrm{d}r\\ &\qquad\qquad = 2\pi\int_0^{\frac{r_\varepsilon}{\varepsilon}}\left(\frac{{f_0}'(t)^2}{f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}+\frac{f_0(t)^2}{s^2 f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}+\frac{1}{2}\left(1-\frac{f_0(t)^2}{f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)^2}\right)^2\right)t\mathrm{d}t,\end{aligned}$$

where in the last equality we used the change of variables $r=\varepsilon t$. Finally, using the fact that $\lim_{\varepsilon\to 0} f_0\left(\frac{r_\varepsilon}{\varepsilon}\right)=1$, which follows from $\lim_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon}=+\infty$, we deduce from (3.1) that

$$\frac{1}{2}\int_{D_s}\left(|\nabla|u_\varepsilon||^2+\rho_\varepsilon^2|Y_\Gamma|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\mathrm{d}v\mathrm{d}w = \left(\pi\log\frac{r_\varepsilon}{\varepsilon}+\gamma+o_\varepsilon(1)\right).$$

By combining this with (3.7) and (3.8), we are led to

$$\begin{aligned}&\frac{1}{2}\int_{T_{r_\varepsilon}^\Omega(\Gamma)}\rho_\varepsilon^2\left(|\nabla u_\varepsilon|^2+\frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\\ &\qquad\leq \frac{1}{2}\left(1+O_\varepsilon(r_\varepsilon)\right)\int_0^{|\Gamma|}\rho_\varepsilon^2(\Gamma(s))\left(\pi\log\frac{r_\varepsilon}{\varepsilon}+\gamma+o_\varepsilon(1)\right)\mathrm{d}s \leq \pi|\rho_\varepsilon^2\Gamma|\frac{r_\varepsilon}{\varepsilon}+|\Gamma|\gamma+o_\varepsilon(1),\end{aligned}$$

where in the last inequality we used that $q>1$, which implies that $O_\varepsilon(r_\varepsilon)\log\frac{r_\varepsilon}{\varepsilon}=o_\varepsilon(1)$. The claim (3.4) is thus proved.

Step 3. Energy estimate outside $T_{r_\varepsilon}^\Omega(\Gamma)$.

In the region $\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}$, we have $|u_\varepsilon|\equiv 1$ and therefore, using (2.5) and (3.2), we find

$$|\nabla_A u_\varepsilon| = |\nabla\varphi - A| = |X_\Gamma + \nabla f_\Gamma - A_\Gamma| = |j_\Gamma|\,.$$

Using once again that $|u_\varepsilon|\equiv 1$ in $\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}$ and that $\rho_\varepsilon\le 1$ in Ω, we have

$$\frac12\int_{\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}}\rho_\varepsilon^2|\nabla_A u_\varepsilon|^2+\frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+\frac12\int_{\mathbb{R}^3}|\operatorname{curl}A|^2\le I_\varepsilon,$$

where

$$I_\varepsilon := \frac12\int_{\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}}|\nabla_A u_\varepsilon|^2+\frac12\int_{\mathbb{R}^3}|\operatorname{curl}A|^2=\frac12\int_{\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}}|j_\Gamma|^2+\frac12\int_{\mathbb{R}^3}|\operatorname{curl}A_\Gamma|^2\,.$$

Recalling (2.6), we obtain

$$I_\varepsilon = C_{\Omega,\Gamma}+\pi|\Gamma|\log r_\varepsilon + o_\varepsilon(1).$$

Hence

$$\frac12\int_{\Omega\setminus\overline{T_{r_\varepsilon}(\Gamma)}}\rho_\varepsilon^2|\nabla_A u_\varepsilon|^2+\frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2+\frac12\int_{\mathbb{R}^3}|\operatorname{curl}A|^2\le \pi|\Gamma|\log r_\varepsilon + C_{\Omega,\Gamma}+o_\varepsilon(1). \tag{3.9}$$

Step 4. Final energy estimate.

By combining the main estimates from the previous steps, namely (3.3), (3.4), and (3.9), we are led to

$$\begin{aligned}F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon,A)&=\frac12\int_\Omega\rho_\varepsilon^2|\nabla_A u_\varepsilon|^2+\frac{\rho_\varepsilon^4}{\varepsilon^2}(1-|u_\varepsilon|^2)^2+\frac12\int_{\mathbb{R}^3}|\operatorname{curl}A|^2\\&\le \pi|\rho_\varepsilon^2\Gamma||\log\varepsilon|+\pi|\Gamma|\log r_\varepsilon + C_{\Omega,\Gamma}+\gamma|\Gamma|+o_\varepsilon(1)\\&\le \pi|\rho_\varepsilon^2\Gamma||\log\varepsilon|+\left(\frac{N}{\alpha}+1\right)\pi|\Gamma|\log|\log\varepsilon|+C_{\Omega,\Gamma}+o_\varepsilon(1),\end{aligned}$$

where, in the last inequality, we fixed the exponent $q=\frac{N}{\alpha}+1>1$ so that $r_\varepsilon=|\log\varepsilon|^{-\left(\frac{N}{\alpha}+1\right)}$, which allows us to absorb the term $\gamma|\Gamma|$. We remark that the choice of q is arbitrary among numbers strictly larger than $\max\{\frac{N}{\alpha},1\}$. This completes the proof of (1.6).

Step 5. Vorticity estimate. Let $B\in C_T^{0,1}(\Omega,\mathbb{R}^3)$. By integration by parts, recalling (1.2) and that $|u_\varepsilon|\equiv 1$ in $\Omega\setminus T_{r_\varepsilon}(\Gamma)$, we have

$$\begin{aligned}\int_\Omega\mu(u_\varepsilon,A)\cdot B&=\int_\Omega\operatorname{curl}\left(j(u_\varepsilon,A)+A\right)\cdot B=\int_\Omega\left(j(u_\varepsilon,A)+A\right)\cdot\operatorname{curl}B\\&=\int_\Omega|u_\varepsilon|^2\nabla\varphi\cdot\operatorname{curl}B+\int_\Omega(1-|u_\varepsilon|^2)A\cdot\operatorname{curl}B\\&=\int_\Omega\nabla\varphi\cdot\operatorname{curl}B+\int_{T^\Omega_{r_\varepsilon}(\Gamma)}(1-|u_\varepsilon|^2)(A-\nabla\varphi)\cdot\operatorname{curl}B.\end{aligned} \tag{3.10}$$

We start by estimating the second integral in the RHS of (3.10). Since $A - \nabla\varphi = j_\Gamma \in L^q(\Omega, \mathbb{R}^3)$ for $q < 2$, using that $1 - |u_\varepsilon|^2 \leq 1$ and $b \leq \rho_\varepsilon^2 \leq 1$, by applying Hölder's inequality we deduce that

$$\begin{aligned}\left|\int_{T_{r_\varepsilon}^\Omega(\Gamma)} (1-|u_\varepsilon|^2) j_\Gamma \cdot \operatorname{curl} B\right| &\leq \|\operatorname{curl} B\|_{L^\infty(\Omega,\mathbb{R}^3)} \|j_\Gamma\|_{L^{\frac{3}{2}}(\Omega,\mathbb{R}^3)} \|(1-|u_\varepsilon|^2)\|_{L^3(T_{r_\varepsilon}^\Omega(\Gamma),\mathbb{R}^3)} \\ &\leq C\|\operatorname{curl} B\|_{L^\infty(\Omega,\mathbb{R}^3)} \|(1-|u_\varepsilon|^2)\|^{\frac{2}{3}}_{L^2(T_{r_\varepsilon}^\Omega(\Gamma),\mathbb{R}^3)} \\ &\leq C\|\operatorname{curl} B\|_{L^\infty(\Omega,\mathbb{R}^3)} \varepsilon^{\frac{2}{3}} \left(\int_\Omega \frac{\rho_\varepsilon^4}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)^{\frac{1}{3}} \\ &\leq C\|\operatorname{curl} B\|_{L^\infty(\Omega,\mathbb{R}^3)} \varepsilon^{\frac{2}{3}} |\log\varepsilon|^{\frac{1}{3}}, \end{aligned} \tag{3.11}$$

where in the last inequality we used that

$$\int_\Omega \frac{\rho_\varepsilon^4}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \leq C|\log\varepsilon|$$

in view of (3.4).

We now proceed to compute the first integral in the RHS of (3.10). Using (3.2) and an integration by parts, we obtain

$$\int_\Omega \nabla\varphi \cdot \operatorname{curl} B = \int_\Omega (X_\Gamma + \nabla f_\Gamma) \cdot \operatorname{curl} B = \int_\Omega \operatorname{curl} X_\Gamma \cdot B. \tag{3.12}$$

Finally, recalling (2.3), from (3.10), (3.11), and (3.12), we deduce that

$$\|\mu(u_\varepsilon, A) - 2\pi\Gamma\|_{(C_T^{0,1}(\Omega,\mathbb{R}^3))^*} \leq C\varepsilon^{\frac{2}{3}} |\log\varepsilon|^{\frac{1}{3}}.$$

This concludes the proof of (1.7) for $\beta = 1$. The proof of the estimate for $\beta \in (0,1)$ follows from interpolation. In fact, from [Rom19b, Lemma 8.1], we have that

$$\|\mu(u_\varepsilon, A)\|_{(C_0^0(\Omega,\mathbb{R}^3))^*} \leq C\left(\frac{1}{2}\int_\Omega |\nabla_A u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + |\operatorname{curl} A|^2\right).$$

Since $\rho_\varepsilon \geq b$, we deduce that the RHS is bounded above by $CF_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A)$. Using (1.6) and $\|\Gamma\|_{(C^0(\Omega))^*} \leq C$, we then deduce that

$$\|\mu(u_\varepsilon, A) - 2\pi\Gamma\|_{(C_0^0(\Omega,\mathbb{R}^3))^*} \leq C|\log\varepsilon|.$$

To conclude, we use interpolation (see, for instance, [JMS04], which builds upon [JS02])

$$\begin{aligned} &\|\mu(u_\varepsilon, A) - 2\pi\Gamma\|_{(C_T^{0,\beta}(\Omega,\mathbb{R}^3))^*} \\ &\qquad \leq \|\mu(u_\varepsilon, A) - 2\pi\Gamma\|^\beta_{(C_T^{0,1}(\Omega,\mathbb{R}^3))^*} \|\mu(u_\varepsilon, A) - 2\pi\Gamma\|^{1-\beta}_{(C_0^0(\Omega,\mathbb{R}^3))^*} \leq C\varepsilon^{\frac{2}{3}\beta}|\log\varepsilon|^{1-\frac{2}{3}\beta}. \end{aligned}$$

This concludes the proof of (1.7) for $\beta \in (0,1)$.

□

4. Upper bound for the first critical field

In this section we provide a proof for Theorem 1.2.

Proof of Theorem 1.2. We consider, for any ε sufficiently small, curves Γ_ε such that (1.8), (1.9), and (1.10) hold. By applying Theorem 1.1 with $\Gamma = \Gamma_\varepsilon$, we obtain a pair $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega,\mathbb{C}) \times H^1(\mathbb{R}^3,\mathbb{R}^3)$ such that

$$\begin{aligned} F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) &= \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{\rho_\varepsilon^4}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A_\varepsilon|^2 \\ &\leq \pi|\rho_\varepsilon^2\Gamma_\varepsilon||\log\varepsilon| + \left(\frac{N}{\alpha}+1\right)\pi \log|\log\varepsilon| + C_{\Omega,\Gamma_\varepsilon} + o_\varepsilon(1) \end{aligned} \tag{4.1}$$

and, for any $\beta \in (0,1]$,

$$\|\mu(u_\varepsilon, A_\varepsilon) - 2\pi\Gamma_\varepsilon\|_{(C_T^{0,\beta}(\Omega,\mathbb{R}^3))^*} \leq C\varepsilon^{\frac{2}{3}\beta}|\log\varepsilon|^{1-\frac{2}{3}\beta}. \tag{4.2}$$

Notice, in particular, that if assumption (1.9) does not hold, then it becomes necessary to carefully track the dependence on the curve length in the constants appearing throughout the proof of Theorem 1.1. The same applies to the present proof, where we will repeatedly use the bound $|\rho_\varepsilon^2\Gamma_\varepsilon| \leq |\Gamma_\varepsilon| \leq C_0$.

We now consider the configuration

$$(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) = (\rho_\varepsilon u_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, A_\varepsilon + h_{\mathrm{ex}} A_\varepsilon^0),$$

where $(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon^0}, h_{\mathrm{ex}}A_\varepsilon^0)$ is the Meissner state. Using the energy splitting (2.7), we find

$$GL_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}}A_\varepsilon^0) + F_{\varepsilon,\rho_\varepsilon}(u_\varepsilon, A_\varepsilon) - h_{\mathrm{ex}}\int_\Omega \mu(u_\varepsilon, A_\varepsilon)\cdot B_\varepsilon^0 + \mathfrak{r}. \tag{4.3}$$

Let us estimate $\mathfrak{r}$. From (2.8), using Hölder's inequality and $b \leq \rho_\varepsilon^2 \leq 1$, we are led to

$$\mathfrak{r} \leq Ch_{\mathrm{ex}}^2 \varepsilon \|\operatorname{curl} B_\varepsilon^0\|_{L^4(\Omega)}^2 \left(\int_\Omega \frac{\rho_\varepsilon^4}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)^{\frac{1}{2}}.$$

From (1.6), we find

$$\int_\Omega \frac{\rho_\varepsilon^4}{\varepsilon^2}(1-|u_\varepsilon|^2)^2 \leq C|\log\varepsilon|.$$

On the other hand, from [DVR25, Proposition 2.2], we have

$$\|\operatorname{curl} B_\varepsilon^0\|_{L^4(\Omega)}^2 \leq C,$$

where C does not depend on ε. Since we assume (1.11), we then deduce that

$$\mathfrak{r} \leq C\varepsilon^{1-2\eta}|\log\varepsilon|^{\frac{1}{2}} = o_\varepsilon(1).$$

Using (4.2), (1.11), and the fact that $B_\varepsilon^0 \in C_T^{0,1}(\Omega, \mathbb{R}^3)$ with $\|B_\varepsilon^0\|_{C_T^{0,\beta}} \leq C$ for any $\beta \in (0,1)$, where the constant C does not depend on ε (see once again [DVR25, Proposition 2.2]), we find

$$h_{\mathrm{ex}} \int_\Omega \mu(u_\varepsilon, A_\varepsilon) \cdot B_\varepsilon^0 = 2\pi h_{\mathrm{ex}} \langle \Gamma_\varepsilon \,, B_\varepsilon^0 \rangle + o_\varepsilon(1) = 2\pi h_{\mathrm{ex}} \mathrm{R}_{\rho_\varepsilon^2}(\Gamma_\varepsilon) |\rho_\varepsilon^2 \Gamma_\varepsilon| + o_\varepsilon(1).$$

Inserting $h_{\mathrm{ex}} \geq H_{c_1}^\varepsilon + K^0 \log|\log\varepsilon| = \frac{|\log\varepsilon|}{2\mathrm{R}_\varepsilon} + K^0 \log|\log\varepsilon|$ and using (1.8), we then obtain

$$h_{\mathrm{ex}} \int_\Omega \mu(u_\varepsilon, A_\varepsilon) \cdot B_\varepsilon^0 \geq \pi |\rho_\varepsilon^2 \Gamma_\varepsilon| |\log\varepsilon| + 2\pi K^0 \mathrm{R}_\varepsilon |\rho_\varepsilon^2 \Gamma_\varepsilon| \log|\log\varepsilon| + O_\varepsilon(1).$$

Using this together with (4.1) in (4.3), we deduce that

$$\begin{aligned} GL_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) &\leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) \\ &\quad + \left(\frac{N}{\alpha} + 1\right) \pi \log|\log\varepsilon| + C_{\Omega,\Gamma_\varepsilon} - 2\pi K^0 \mathrm{R}_\varepsilon |\rho_\varepsilon^2 \Gamma_\varepsilon| \log|\log\varepsilon| + O_\varepsilon(1). \end{aligned} \tag{4.4}$$

Let us now show that there exists $c_0 > 0$ (independent of ε) such that

$$|\rho_\varepsilon^2 \Gamma_\varepsilon| \geq c_0 > 0. \tag{4.5}$$

We start by closing Γ_ε by connecting its endpoints with a curve lying on $\partial\Omega$. More precisely, we consider an arbitrary smooth curve on $\partial\Omega$ that connects the endpoints of Γ_ε, oriented consistently with the orientation of Γ_ε and such that, if we denote by $\tilde{\Gamma}_\varepsilon$ the resulting loop, we have $|\tilde{\Gamma}_\varepsilon| \leq C|\Gamma_\varepsilon|$. Since $B_\varepsilon^0 \times \nu = 0$ on $\partial\Omega$, by Stokes' theorem, we have

$$\langle \Gamma_\varepsilon \,, B_\varepsilon^0 \rangle = \langle \tilde{\Gamma}_\varepsilon \,, B_\varepsilon^0 \rangle = \int_{S_{\Gamma_\varepsilon}} \operatorname{curl} B_\varepsilon^0,$$

where S_{Γ_ε} denotes a surface with least area among those whose boundary is $\tilde{\Gamma}_\varepsilon$, i.e. a solution to the associated Plateau's problem.

By Hölder's inequality and the isoperimetric inequality, we have

$$\int_{S_{\Gamma_\varepsilon}} |\operatorname{curl} B_\varepsilon^0| \leq \|\operatorname{curl} B_\varepsilon^0\|_{L^4(\Omega,\mathbb{R}^3)} \mathrm{Area}(S_{\Gamma_\varepsilon})^{\frac{3}{4}} \leq C(|\tilde{\Gamma}_\varepsilon|^2)^{\frac{3}{4}} \leq C|\Gamma_\varepsilon|^{\frac{3}{2}} \leq C|\rho_\varepsilon^2 \Gamma_\varepsilon|^{\frac{3}{2}}.$$

Therefore,

$$\mathrm{R}_{\rho_\varepsilon^2}(\Gamma_\varepsilon) \leq C|\rho_\varepsilon^2 \Gamma_\varepsilon|^{\frac{1}{2}},$$

which combined with $\liminf\limits_{\varepsilon\to 0} \mathrm{R}_\varepsilon > 0$ and (1.8) yields the claim.

By combining (4.5) and (1.10) with (4.4), we are led to

$$\begin{aligned} GL_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) &\leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) \\ &\quad + \left(\left(\frac{N}{\alpha} + 1\right)\pi + C_0 - 2\pi c_0 K^0 \frac{1}{2} \liminf_{\varepsilon\to 0} \mathrm{R}_\varepsilon\right) \log|\log\varepsilon| + O_\varepsilon(1), \end{aligned}$$

where we also used that $R_\varepsilon \geq \frac{1}{2} \liminf_{\varepsilon\to 0} > 0$ for any ε sufficiently small.

Hence, provided

$$K^0 > \frac{\left(\frac{N}{\alpha}+1\right)\pi + C_0}{\pi c_0 \liminf\limits_{\varepsilon\to 0} \mathrm{R}_\varepsilon} + 1,$$

we have

$$GL_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) - \log|\log\varepsilon| + O_\varepsilon(1). \tag{4.6}$$

On the other hand, by [DVR25, Theorem 1.2], since we assume $h_{\mathrm{ex}} \leq \varepsilon^{-\eta}$, for $\eta \in \left(0, \frac{1}{2}\right)$ (recall (1.11)), we know that if $(\mathbf{u}, \mathbf{A})$ is a configuration without vortex lines — that is, $|\mathbf{u}| > c > 0$ for some $c > 0$ — such that

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) \leq GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0),$$

then necessarily

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) = GL_\varepsilon(\rho_\varepsilon e^{ih_{\mathrm{ex}}\phi_\varepsilon}, h_{\mathrm{ex}} A_\varepsilon^0) + o_\varepsilon(1).$$

Combining this with (4.6), yields

$$GL_\varepsilon(\mathbf{u}, \mathbf{A}) > GL_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon) + \frac{1}{2}\log|\log\varepsilon|.$$

We have thus constructed a configuration $(\mathbf{u}_\varepsilon, \mathbf{A}_\varepsilon)$ with a single vortex line located at Γ_ε, such that its energy is strictly less than any configuration without vortex lines. Hence the global minimizer of the energy GL_ε must have vortex lines provided

$$h_{\mathrm{ex}} \geq H_{c_1}^\varepsilon + K^0 \log|\log\varepsilon|.$$

This concludes the proof. □

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